# ANSWERING YOUR STUDENTS' WHY QUESTIONS IN MATHEMATICS

- Why can't we divide by zero?
- Why is a negative times a negative a positive?
- Why is $n^0 = 1$?

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# Why can't a number be divided by zero?

## A real-world approach

Ask six students to stand in a line in front of class. Then describe characteristics of them using fractions. For example:
1/6 of them have blond hair.
1/3 of them are wearing shorts.

Ask the students to come up with other fractional descriptions. Examples might include:
1/2 of them are girls.
2/3 of them have brown eyes.
5/6 of them are wearing sweatshirts.
6/6 of them think the teacher is groovy.

Now look at the variety of fractions we have created. There are many different numerators and denominators. Can you suggest a fraction with a numerator of zero?
0/6 of them have feathers.

However, can you think of a fraction that would have a denominator of zero? There's no sensible way to express such a fraction. It has no meaning. By using different numbers of students, we can generate other denominators, but we can't get a denominator of zero.

The problem with this model is that you cannot model a mixed number or an improper fraction. You cannot write a fraction in which the numerator is greater in value than the denominator, yet such fractions are not undefined.

## A pattern approach (situated in a real-world context)

Let 12 ÷ 6 mean that we remove 6 piles of sand each day. Then the quotient, 2, refers to how many days it would take to remove the 12 piles of sand. Or, put another way, 12 - 6 - 6 = 0.

12 ÷ 3 = 4 (or, 12 - 3 - 3 - 3 - 3 = 0) means that if we remove 3 piles of sand each day, then 12 piles will be removed in 4 days.

12 ÷ 1/2 = 24 (or 12 - 1/2 - 1/2 - 1/2 - 1/2 - 12 - ... - 1/2 = 0 means that if we remove ½ a pile of sand every day, then it will take 24 days to remove 12 piles.

12 ÷ 1/4 = 48 (or 12 - 1/4 - 1/4 - 1/4 - 1/4 ... - 1/4 = 0 after we subtract 48 fourths.)
12 ÷ 1/1,000 = 12,000 (or 12 - 1/1,000 - .... - 1/1,000 = 0 after we subtract 12,000 one-thousandths.)

Clearly, our landscaping company would be out of business if we only removed 1/1,000 of a pile of sand each day. However, with mathematics, imagining the scenario is enough, and we see it would take 12,000 days.

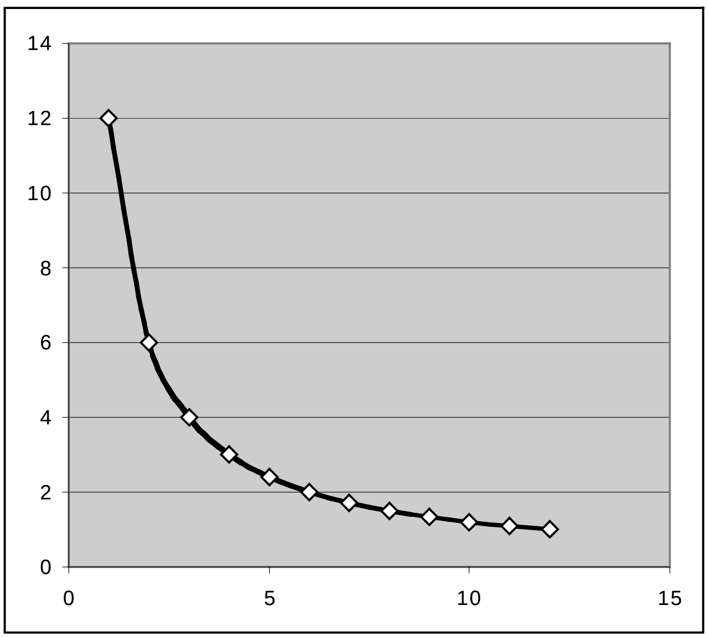
Even if we removed one grain of sand each day, if we had enough time, we would eventually move all the sand.

Now, consider 12 ÷ 0. This means that we remove 0 piles of sand a day. Or, looking at the subtraction, we would have 12 - 0 - 0 - 0 - 0 - 0 - 0 - 0 .....

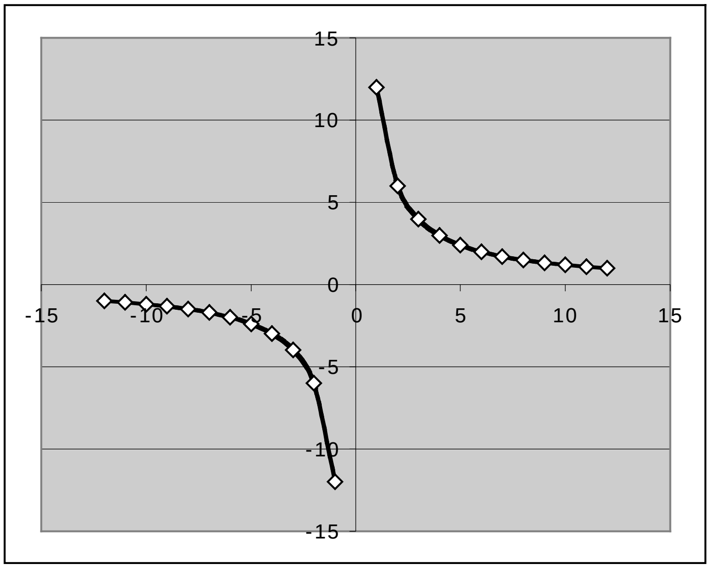
So how many 0s must I subtract from 12 to get to 0? There is no number of zeros, and that is why 12 ÷ 0 is undefined.

## A graphical approach

Think of the fraction 12/12. It is equal to one. Decrease the divisor and the quotient increases. 12/6 = 2. 12/4 = 3. 12/3 = 4. When the divisor, x, is less than one but greater than zero, we get very large values of 12/x. For example, 12/.01 = 1,200. As the divisor approaches zero, then 12/x approaches infinity. We might assume that n/0 = infinity. This is shown in the graph.




However, we can also approach zero from the negative end of the number line. 12/($^{-}$12) = $^{-}$1. 12/($^{-}$4) = $^{-}$3, and 12/($^{-}$1) = $^{-}$12. Thus as we approach zero from the negative side of the number line, the quotient of n/0 approaches $^{-}$infinity. As shown in the second graph, this also leads to a contradiction as we get two divergent values for n/0.




## A proof by negation

Let's use a proof by negation to demonstrate why we cannot divide by zero.
If we can divide by zero, then the following results.
1 x 0 = 0
2 x 0 = 0
By the transitive property, 1 x 0 = 2 x 0.
Dividing by zero results in this contradiction:  1 = 2

Since we know that this is a false statement, our original assumption that we can divide by zero must be wrong.

# Why is a negative times a negative positive?

## A real-world approach

Modeling the product of two negatives requires a situation involving two quantities, both of which have two directions. I will consider delivering mail by a postal deliverer who makes many mistakes. We can receive (+) a check (+) or a bill (-), and if we receive a check or a bill by mistake, then we may have to return (-) a check (+) or a bill (-).

If we *receive* **(+)** *checks* **(+),** then we have more **(+)** money.

If we *receive* **(+)** *bills* **(-),** then we have less **(-)** money.

If we were erroneously sent checks then we have to *return* **(-)** *checks* **(+)**, and we have less **(-)** money.

If we were erroneously sent bills, then we have to *return* **(-)** *bills* **(-),** and we have more **(+)**money.

## A pattern approach

Consider the following table showing the products of a positive number multiplied by a second number that is decreasing in value:

3 x 3 = 9
3 x 2 = 6
3 x 1 = 3

Notice that as the second factor decreases by one, the product decreases by three. It is moving *down* the number line in steps of three. Continuing this pattern gives us:

3 x 0 = 0
3 x -1 = -3
3 x -2 = -6
3 x -3 = -9

Because your brain detects patterns so well, this is a very reasonable way to see that when positive and negative numbers are multiplied, a negative number results.

Now lets take the last multiplication fact and begin decreasing the lead factor by one.

3 x -3 = -9
2 x -3 = -6
1 x -3 = -3

Now the product is advancing *up* the number line in increments of three. The product is increasing. If we continue this pattern the following table results:

0 x -3 = 0
-1 x -3 = 3
-2 x -3 = 6
-3 x -3 = 9

Since our brain trusts patterns, the student is confident in the resulting conclusion that a negative multiplied by a negative yields a positive product.

## A number sense approach

Let's consider the problem 2 x 3 which we know to be equal to 6. One way to think about negative numbers is as inverses of positive numbers. In fact, every non-zero number has an inverse, and the sum of a number and it's inverse is zero. Thus 5 + $^{-}5$ = 0. That means that we could read the following problem as, "What is the inverse of the product of two and three?"
$^{-}2$ x 3 = ?

Since 2 x 3 is six, then the inverse of this must be $^{-}6$. Thus:
$^{-}2$ x 3 = $^{-}6$

Now we have demonstrated by example that when a negative is multiplied by a positive, the product is negative. Since the Commutative Property applies to multiplication, we can rewrite the problem this way:
3 x $^{-}2$ = $^{-}6$

Next let's ask ourselves, "What is the inverse of 3 x $^{-}2$ = $^{-}6$?" Clearly it would be:
$^{-}3$ x $^{-}2$ = ?

Since we know that the answer to the previous problem was $^{-}6$, the answer to the inverse problem should be the inverse of $^{-}6$, which is $^{+}6$. Thus we have a demonstration that a negative multiplied by a negative is positive.

# Why is $n^0 = 1$?

## A real-world approach

Take out a piece of paper and fold it in half as many times as possible. You will probably only achieve about six or seven folds. Why is this? It is because the number of layers is doubling with every fold. That is, they are growing exponentially. In fact, we can use exponents to study the process. Let $2^n$ represent *n* doublings or folds of the paper. Then the following t-table shows what is happening to the number of layers as we fold the paper in half each time.

| folds | layers |
|---|---|
| $2^1$ | 2 |
| $2^2$ | 4 |
| $2^3$ | 8 |
| $2^4$ | 16 |
| $2^5$ | 32 |
| $2^6$ | 64 |
| $2^7$ | 128 |

In each row, the base indicates that the fold *doubles* the number of layers. The exponent tells how many total folds have been made. The right side of the table shows the total number of resulting layers. With this understanding, we must ask ourselves what is meant by the following notation?

$$2^0$$

It has to mean that the piece of paper has not been folded (not been doubled). But how many layers do we have in this case? Clearly $2^0 = 1$. If $2^0$ was equal to zero as is often suspected by students, then the assignment would have been to get out *no* piece of paper and then fold it.

| folds | layers |
|---|---|
| $\mathbf{2^0}$ | **1** |
| $2^1$ | 2 |
| $2^2$ | 4 |
| $2^3$ | 8 |
| $2^4$ | 16 |
| $2^5$ | 32 |
| $2^6$ | 64 |
| $2^7$ | 128 |

Similar arguments can be made regarding tripling or quadrupling folds. No matter what *type* of fold you make, you always begin with one layer of paper. Thus $n^0 = 1$ for all non-zero values of *n*.

## A pattern approach

We know that the following powers of two are correct:
$2^2 = 4$
$2^3 = 8$
$2^4 = 16$
$2^5 = 32$

If we explore this pattern in reverse we see that each answer is one half of the previous answer.
$2^5 = 32$
$2^4 = 16$
$2^3 = 8$
$2^2 = 4$
$2^1 = 2$

So if we continue the pattern, then we should have

$2^0 = 1$.

We could even extend this pattern into negative powers to get fractions:
$2^{-1} = 1/2$
$2^{-2}= 1/4$
$2^{-3} = 1/8$

While this patterning is a comfortable mode of operation for our brain, exponents are so abstract to most young learners that even this method may not adequately convince them that any non-zero number to the zero power is one.

## A symbolic approach

The laws of exponents and the properties of algebra lead us to this proof that $n^0 = 1$ for all non-zero values of n.

| | |
|---|---|
| $n^a/n^a = 1$ | Any non-zero number divided by itself is equal to 1 |
| $n^a/n^a = n^{a-a}$ | Laws of exponents |
| $n^{a-a} = n^0$ | Any number and its inverse equals zero |
| $n^0 = 1$ | Transitive property |

The problem with this concise proof is that the reader must understand deductive reasoning, the transitive property, and the laws of exponents to understand that $n^0 = 1$. If the student is struggling with $n^0$ then they probably are struggling even more with these more advanced concepts.